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An Inverse Problem in Birth and Death Processes

P. R. PARTHASARATHY AND R. B. LENIN
Department of Mathematics
Indian Institute of Technology, Madras, Chennai 600 036, India

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Abstract—An inverse problem of constructing birth and death processes $\{X(t)\}$ on finite state space $\{0, 1, 2, \ldots, N\}$ is considered. Given a set of $2N + 1$, distinct, nonnegative real numbers one of which is zero, say

$$0 = s_0 < z_1 < s_2 < \cdots < z_N < s_N,$$

a procedure is established to obtain the birth and death rates of a birth and death process so that

$$P(X(t) = 0) = \sum_{j=0}^{N} \frac{\prod_{i=1}^{N} (z_i - s_j)}{\prod_{i=0, i \neq j}^{N} (s_i - s_j)} e^{-s_j t}$$

and other transient system size probabilities. This technique is illustrated numerically.

Keywords—Tridiagonal matrices, Tridiagonal determinants, Persymmetric matrix, Eigenvalues, Orthogonal polynomials.

1. INTRODUCTION

Recurrence relations are important mathematical tools of computation. There is hardly a computational task which does not rely on recursive techniques at one time or another. The widespread use of recurrence relations can be ascribed to their intrinsic constructive quality and the great ease with which they are amenable to mechanisation. In particular, the three-term recurrence relations lie at the heart of Continued Fractions (CFs), Orthogonal Polynomials (OPs), and Birth and Death Processes (BDPs) [1–3].

There is an intimate relationship between three-term recurrence relations and tridiagonal matrices. Tridiagonal matrices appear frequently in various branches of mathematics, and their spectral properties and inverse problems have been studied extensively. Historical remarks and physical motivations for these problems can be found in [4–6]. Linear systems occurring frequently in the numerical solution of partial and coupled ordinary differential equations have coefficient

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matrices with certain band structures and among these, systems having tridiagonal matrices are encountered often.

BDPs have applications to a variety of fields including optics, epidemics, queueing and inventory models, neutron propagation, chemical reactions, population dynamics, genetic models, and production management, to mention a few (e.g., [7–12]). In the study of birth and death processes, most of the results have been limited to steady state solutions because the equations involved are simple, and relatively straightforward techniques can be employed. Steady state results are, however, inappropriate in many applied situations when the time horizon of operations is finite. In many potential applications, the practitioner needs to know how the system will operate up to some specified time. There has been a resurgence of interest in the literature in the study of transient analysis of birth and death processes [13–16].

BDPs on a finite state space cover a large spectrum of operations research and biological systems. Queueing models with finite state space have applications in production and inventory problems, e.g., to optimize the size of the storage space, to determine the trade-off between throughput and inventory (or waiting time), and to exhibit the propagation of blockage. The performance of the produce-to-stock manufacturing facility can be obtained from the performance of the finite queueing systems [17]. The situation of a network of queues with finite buffers occurs widely in computer and telecommunications systems [18]. Stochastic population models like logistic and power law involve finite BDPs [19].

The infinitesimal matrix of a BDP is tridiagonal with real and distinct eigenvalues [20,21]. Therefore, the transient system size probabilities can be expressed as a sum of exponentials which occurs frequently in the literature [22]. Many authors have used distribution functions related to the exponential stages formulation introduced by Erlang. Much of the current popularity of such distributions is due to the pioneering work of Neuts and his colleagues on the phase-type family exploiting relationships to Markov chains and adopting the theory to effective computational use in a variety of stochastic models [23].

The purpose of this paper is to study an inverse problem of constructing BDPs on finite state space $\{0, 1, \dots, N\}$. Given a set of $2N + 1$, distinct, nonnegative real numbers of which one is zero, say

$$0 = s_0 < z_1 < s_1 < \cdots < z_N < s_N, \tag{1.1}$$

a procedure is established to obtain the birth and death rates of a BDP $\{X(t)\}$ on finite state space for which

$$P(X(t) = 0) = p_0(t) = \sum_{k=0}^{N} w_{0k} e^{-s_j t}, \tag{1.2}$$

where

$$w_{0k} = \frac{\prod_{i=1}^{N} (z_i - s_k)}{\prod_{i=0, i\neq k}^{N} (s_i - s_k)}.$$

Obviously, $w_{0k} > 0$ and $\sum_{k=0}^{N} w_{0k} = 1$. This procedure uses the method of de Boor and Golub [24] for constructing tridiagonal symmetric matrices.

Alternatively, given $N+1$ distinct real numbers of which one is zero, say $0 = s_0 < s_1 < \cdots < s_N$, and positive real weights $w_{00}, w_{01}, \dots, w_{0N}$ such that $\sum_{k=0}^{N} w_{0k} = 1$, a finite BDP is constructed for which (1.2) is true. Once the birth and death rates are identified, the remaining system size time-dependent probabilities for the associated birth and death process can be obtained. These are illustrated numerically.

2. DIRECT PROCEDURE

Consider a finite BDP $\{X(t), t \geq 0\}$ with states $\{0, 1, \ldots, N\}$. Let $\lambda_n \Delta t + o(t)$, $n = 0, 1, \ldots, N$ be the probability of a birth occurring in $(t, t + \Delta t]$ and $\mu_n \Delta t + o(t)$, $n = 1, 2, \ldots, N$ be the probability of a death occurring in $(t, t + \Delta t]$ when there are n units in the system. Let $p_r(t) = P(X(t) = r)$, $r = 0, 1, \ldots, N$ be the probability that there are r units in the system at time t. Then these probabilities satisfy the following set of finite differential equations:

$$\frac{d}{dt} p_0(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t),$$
$$\frac{d}{dt} p_r(t) = \lambda_{r-1} p_{r-1}(t) - (\lambda_r + \mu_r) p_r(t) + \mu_{r+1} p_{r+1}(t), \qquad r = 1, 2, \ldots, N-1,$$

and

$$\frac{d}{dt} p_N(t) = \lambda_{N-1} p_{N-1}(t) - \mu_N p_N(t),$$

such that $0 \leq p_r(t) \leq 1$, $\sum_{r=0}^{N} p_r(t) = 1$ and $p_r(0) = \delta_{rm}$, m—the initial state.

Assuming $P(X(0) = 0) = 1$, these system size probabilities are given by

$$P_n(t) = \sum_{k=0}^{N} w_{nk} e^{-s_k t}, \qquad n = 0, 1, \ldots, N, \tag{2.1}$$

where $-s_0, -s_1, \ldots, -s_N$ are the roots of the tridiagonal determinant

$$B_{N+1}(u) = D(u, \lambda_0, \lambda_1, \ldots, \lambda_{N-1}; \mu_1, \mu_2, \ldots, \mu_N)$$
$$= \begin{vmatrix} u+\lambda_0 & -\mu_1 & & & & \\ -\lambda_0 & u+\lambda_1+\mu_1 & -\mu_2 & & & \\ & -\lambda_1 & u+\lambda_2+\mu_2 & & & \\ & & & \ddots & & \\ & & & & u+\lambda_{N-1}+\mu_{N-1} & -\mu_N \\ & & & & -\lambda_{N-1} & u+\mu_N \end{vmatrix}_{(N+1)\times(N+1)}. \tag{2.2}$$

Here,

$$w_{nk} = \frac{B_n(-s_k)}{B'_{N+1}(-s_k)}, \qquad n = 0, 1, \ldots, N; \quad k = 0, 1, \ldots, N,$$

$$B_n(u) = \left(\prod_{j=0}^{n-1} \lambda_j \right) D(u, \lambda_{n+1}, \ldots, \lambda_{N-1}; \mu_{n+1}, \ldots, \mu_N), \qquad \text{for } n = 1, 2, \ldots, N-2,$$
$$B_{N-1}(u) = \left(\prod_{j=0}^{N-2} \lambda_j \right) (u + \mu_N),$$
$$B_N(u) = \prod_{j=0}^{N-1} \lambda_j,$$

and $B_0(u)$ is the tridiagonal determinant obtained from $B_{N+1}(u)$ by deleting its first row and first column [25]. This can be proved by writing Laplace transforms of these probabilities as a continued fraction and finding the inversion using partial fractions. More details can be had from [14,26].

We observe that $B_{N+1}(u)$ given by (2.2) is clearly zero when $-s$ is an eigenvalue of the matrix

$$\begin{bmatrix} \lambda_0 & \mu_1 & & & & \\ \lambda_0 & \lambda_1+\mu_1 & \mu_2 & & & \\ & \lambda_1 & \lambda_2+\mu_2 & \mu_3 & & \\ & & & \ddots & & \\ & & & & \lambda_{N-1} & \mu_N \end{bmatrix}_{(N+1)\times(N+1)}.$$

This matrix is transformed into a real symmetric tridiagonal matrix

$$E = \begin{bmatrix} \lambda_0 & \sqrt{\lambda_0\mu_1} & & & & \\ \sqrt{\lambda_0\mu_1} & \lambda_1+\mu_1 & \sqrt{\lambda_1\mu_2} & & & \\ & \sqrt{\lambda_1\mu_2} & \lambda_2+\mu_2 & \sqrt{\lambda_2\mu_3} & & \\ & & & \ddots & & \\ & & & & \sqrt{\lambda_{N-1}\mu_N} & \mu_N \end{bmatrix}_{(N+1)\times(N+1)} \tag{2.3}$$

by a similarity transformation.

This is a real symmetric positive definite tridiagonal matrix with nonzero subdiagonal elements, and therefore, the eigenvalues are real and distinct [27] and we denote these eigenvalues by $s_0, s_1, \ldots, s_N$, i.e., $-s_0, -s_1, \ldots, -s_N$ are the roots of $B_{N+1}(u)$ with $s_0 = 0$ for the reason discussed below. Similarly, the roots of $B_0(u)$ are negative, real, and distinct and suppose $-z_1, -z_2, \ldots, -z_N$ are its roots. It is well known that the sequence of the two sets of eigenvalues $(s_k)_0^N$ and $(z_k)_1^N$ interlace so that

$$0 = s_0 < z_1 < s_1 < \cdots < z_N < s_N. \tag{2.4}$$

Since $B'_{N+1}(-s_j) = \prod_{k=0,k\neq j}^N (s_k - s_j)$ and $B_0(-s_j) = \prod_{k=1}^N (z_k - s_j)$, from (2.1) $p_0(t)$ can be written as

$$p_0(t) = \sum_{k=0}^N \frac{\prod_{i=1}^N (z_i - s_k)}{\prod_{i=0,i\neq k}^N (s_i - s_k)} e^{-s_k t}. \tag{2.5}$$

A finite BDP with $\mu_0 = \lambda_N = 0$ is ergodic and the associated tridiagonal rate matrix is conservative, and hence, one of its eigenvalues is 0 [28]. Alternatively, using (2.2) and the following two interesting identities of tridiagonal determinants:

$$\begin{vmatrix} a_1 & b_1 & \cdot & \cdots \\ c_1 & a_2 & b_2 & \cdots \\ \cdot & c_2 & a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = \begin{vmatrix} a_1 & 1 & \cdot & \cdots \\ b_1c_1 & a_2 & 1 & \cdots \\ \cdot & b_2c_2 & a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix}$$

and

$$\begin{vmatrix} A+a_0 & a_0 & & & & & \\ b_1 & A+a_1+b_1 & a_1 & & & & \\ & b_2 & A+a_2+b_2 & a_2 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & b_{n-1} & A+a_{n-1}+b_{n-1} & a_{n-1} \\ & & & & & b_n & A+b_n \end{vmatrix}_{(n+1)\times(n+1)}$$

$$= A \times \begin{vmatrix} A+a_0+b_1 & b_1 & & & & \\ a_1 & A+a_1+b_2 & b_2 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & a_{n-2} & A+a_{n-2}+b_{n-1} & b_{n-1} \\ & & & & b_{n-1} & A+a_{n-1}+b_n \end{vmatrix}_{n\times n},$$

we find that

$$B_{N+1}(u) = u \times \begin{vmatrix} u+\lambda_0+\mu_1 & \lambda_1 & & & & \\ \mu_1 & u+\lambda_1+\mu_2 & \lambda_2 & & & \\ & \mu_2 & u+\lambda_2+\mu_3 & \lambda_3 & & \\ & & & \ddots & & \\ & & & & \mu_{N-1} & u+\lambda_{N-1}+\mu_N \end{vmatrix}_{N\times N}.$$

This indicates that one of the roots of $B_{N+1}(u)$ is 0. Hence, the steady state probabilities p_n, $n = 0, 1, \dots, N$ exist and are given by

$$p_n = w_{n0} = \frac{\left(\prod_{k=n+1}^{N} \mu_k\right)\left(\prod_{k=0}^{n-1} \lambda_k\right)}{\prod_{k=1}^{N} s_k}, \qquad n = 0, 1, \dots, N.$$

3. INVERSE PROCEDURE

In this section, we consider the inverse problem of finding birth and death rates $(\lambda_k)_0^{N-1}$ and $(\mu_k)_1^N$, respectively, from the knowledge of spectral data $(s_k)_0^N$ and $(z_k)_1^N$ satisfying (1.1). This is achieved by constructing a real, symmetric, tridiagonal matrix E given by (2.3), by suitably modifying the method of de Boor and Golub [24].

The elements λ_i and μ_i are computed concurrently by

$$\lambda_0 = \frac{\sum_{k=0}^{N} s_k w_{0k}}{\sum_{k=0}^{N} w_{0k}},$$
$$\lambda_{i-1}\mu_i := \frac{\langle q_i, q_i\rangle}{\langle q_{i-1}, q_{i-1}\rangle}, \qquad i = 1, 2, \dots, N, \tag{3.1}$$

and

$$\lambda_i + \mu_i = \frac{\langle tq_i, q_i\rangle}{\langle q_i, q_i\rangle}, \qquad i = 1, 2, \dots, N.$$

Here q_n is a monic polynomial of degree n recursively given by

$$\begin{aligned} q_{-1}(t) = 0, \qquad q_0(t) = 1 \\ q_{n+1}(t) = (t - \lambda_n - \mu_n)q_n(t) - \lambda_{i-1}\mu_i q_{i-1}(t), \qquad n = 0, 1, 2, \dots, N, \end{aligned} \tag{3.2}$$

$$\langle q_i, q_i\rangle := \sum_{k=0}^{N} q_i(s_k) q_i(s_k) w_{0k}$$

and

$$w_{0k} = \frac{\prod_{i=1}^{N} (z_i - s_k)}{\prod_{i=0, i\neq k}^{N} (s_i - s_k)} = \frac{q_N(s_k)}{q'_{N+1}(s_k)}.$$

Table 1. Computed birth and death rates from $(s_k)_0^N$ and $(z_k)_1^N$.

k	s_{k-1}	z_k	w_{0k-1}	λ_{k-1}	μ_k
1	0.0000	0.0501	0.1685	0.4950	0.2596
2	0.1254	0.2332	0.1642	0.2206	0.3320
3	0.3063	0.3510	0.0592	0.2388	0.2677
4	0.4121	0.4644	0.0660	0.1582	0.3543
5	0.5133	0.5297	0.0250	0.1136	0.5063
6	0.5911	0.6295	0.0852	0.0582	0.5208
7	0.6885	0.7254	0.1313	0.0441	0.5232
8	0.7362	0.7615	0.0372	0.0304	0.6553
9	0.7702	0.8278	0.0290	0.0170	0.6381
10	0.8460	0.8682	0.0238	0.0221	0.6751
11	0.8886	0.9410	0.0436	0.0014	0.7452
12	0.9995		0.1670		

Table 2. Transient system size probability values, mean, and variance at different time points where (k), $m(t)$, and $v(t)$ denote, respectively, 10^k, the mean, and the variance at different time points.

Time (t)	0	1	50	100	150	151
$p_0(t)$	1.000000	6.495695(−1)	1.687827(−1)	1.684721(−1)	1.684715(−1)	1.684715(−1)
$p_1(t)$	0.000000	3.142661(−1)	3.216845(−1)	3.212422(−1)	3.212413(−1)	3.212413(−1)
$p_2(t)$	0.000000	3.335638(−2)	2.134566(−1)	2.134470(−1)	2.134470(−1)	2.134470(−1)
$p_3(t)$	0.000000	2.698729(−3)	1.900707(−1)	1.904193(−1)	1.904200(−1)	1.904200(−1)
$p_4(t)$	0.000000	1.068788(−4)	8.471975(−2)	8.502188(−2)	8.502246(−2)	8.502246(−2)
$p_5(t)$	0.000000	2.390916(−6)	1.898209(−2)	1.907757(−2)	1.907775(−2)	1.907775(−2)
$p_6(t)$	0.000000	2.293055(−8)	2.117020(−3)	2.131624(−3)	2.131652(−3)	2.131652(−3)
$p_7(t)$	0.000000	1.435731(−10)	1.781465(−4)	1.797971(−4)	1.798003(−4)	1.798003(−4)
$p_8(t)$	0.000000	5.358239(−13)	8.249368(−6)	8.344863(−6)	8.345044(−6)	8.345044(−6)
$p_9(t)$	0.000000	9.961163(−16)	2.185983(−7)	2.217869(−7)	2.217929(−7)	2.217929(−7)
$p_{10}(t)$	0.000000	2.172359(−18)	7.127715(−9)	7.255815(−9)	7.256058(−9)	7.256058(−9)
$p_{11}(t)$	0.000000	2.730871(−22)	1.339236(−11)	1.368258(−11)	1.368313(−11)	1.368313(−11)
$m(t)$	0.000000	3.895147(−1)	1.766616	1.768987	1.768991	1.768991
$v(t)$	0.000000	3.220292(−1)	1.680770	1.682881	1.682885	1.682885

This construction leads to a unique tridiagonal symmetric matrix [29]. Using these birth and death rates, the transient system size probabilities are calculated using (2.1). Observe that these w_{0k}'s are the same as the one defined in the previous section.

Knowledge of s_k's alone will not uniquely determine birth and death rates of a BDP. For example, two different birth and death processes with the following birth and death rates given by

$$\lambda_n = (N-n)\alpha, \quad n = 0, 1, \dots, N-1; \qquad \mu_n = n\beta, \quad n = 1, 2, \dots, N \tag{3.3}$$

and

$$\begin{aligned} \lambda_n &= \frac{(N-n)(n+1)}{2(2n+1)}(\alpha+\beta), \qquad n = 0, 1, \dots, N-1, \\ \mu_n &= \frac{n(N+n+1)}{2(2n+1)}(\alpha+\beta), \qquad n = 1, 2, \dots, N, \end{aligned} \tag{3.4}$$

have the same eigenvalues $0, (\alpha+\beta), 2(\alpha+\beta), \dots, (N-1)(\alpha+\beta)$ [14].

We observe that the matrix E given by (2.3) associated with the BDP with rates given by (3.3) is persymmetric when $\alpha = \beta$, that is, it satisfies the following property:

$$\begin{aligned} \lambda_0 &= \mu_N, \\ \lambda_i + \mu_i &= \lambda_{N-i} + \mu_{N-i}, \qquad i = 1, 2, \dots, N-1, \end{aligned} \tag{3.5}$$

and

$$\lambda_{i-1}\mu_i = \lambda_{N-i}\mu_{N+1-i}, \qquad i = 1, 2, \dots, N.$$

Hochstadt [29] shows that the spectrum $(s_i)_0^N$ is enough to determine E uniquely and in this case the weights w_{0k}'s in (3.1) are given by

$$w_{0k} = \frac{1}{\prod_{i=0, i \neq k}^{N} |s_i - s_k|}, \qquad k = 0, 1, \dots, N.$$

REMARK. Any polynomial sequence that satisfies a recurrence of the form (3.2) is an orthogonal polynomial system [30]. The above inverse problem for BDPs has the following orthogonal polynomial interpretation: given $2N + 1$ data $(s_k)_0^N$ and $(z_k)_1^N$ with condition (1.1), the above procedure helps us to construct a finite set OPs $\{q_n(t)\}$ such that s_k's are the roots of $q_{N+1}(t)$ and $(z_k)_1^N$ are the roots of $q_N(t)$.

4. NUMERICAL ILLUSTRATION

In Table 1, values of given spectral data, computed birth and death rates, and weights w_{0k} are presented for the case $N = 11$. Observe that $w_{0k} > 0$ and $\sum_{k=0}^{N} w_{0k} = 1$. The corresponding transient system size probabilities, mean, and variance are tabulated in Table 2. We observe that around 150 time units steady state is reached.

Table 3. Computed birth and death rates from $(s_k)_0^N$ and binomial weights $(w_{0k})_0^N$ where (k) denotes 10^k.

k	s_{k-1}	w_{0k-1}	λ_{k-1}	μ_k
1	0.0000	1.1529(−2)	4.0000	0.8000
2	1.0000	5.7646(−2)	3.8000	1.6000
3	2.0000	1.3691(−1)	3.6000	2.4000
4	3.0000	2.0536(−1)	3.4000	3.2000
5	4.0000	2.1820(−1)	3.2000	4.0000
6	5.0000	1.7456(−1)	3.0000	4.8000
7	6.0000	1.0910(−1)	2.8000	5.6000
8	7.0000	5.4550(−2)	2.6000	6.4000
9	8.0000	2.2161(−2)	2.4000	7.2000
10	9.0000	7.3870(−3)	2.2000	8.0000
11	10.0000	2.0314(−3)	2.0000	8.8000
12	11.0000	4.6168(−4)	1.8000	9.6000
13	12.0000	8.6566(−5)	1.6000	10.4000
14	13.0000	1.3318(−5)	1.4000	11.2000
15	14.0000	1.6647(−6)	1.2000	12.0000
16	15.0000	1.6647(−7)	1.0000	12.8000
17	16.0000	1.3006(−8)	0.8000	13.6000
18	17.0000	7.6504(−10)	0.6000	14.4000
19	18.0000	3.1877(−11)	0.4000	15.2001
20	19.0000	8.3886(−13)	0.1999	16.0000
21	20.0000	1.0486(−14)		

In Table 3, values of gives one spectral data $(s_k)_0^N$, weights $(w_k)_0^N$ and computed birth and death rates are presented. The weights w_k's are binomial probabilities with parameters $N = 20$ and $p = 0.2$.

REMARK. Computationally, the method presented in Section 2 gives a reasonably good approximation for the steady state for a larger N, though it may take a longer execution time. However, for some values of the parameter, the diagonal elements become large making the eigenvalues large and thus causing overflow while running the program.

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